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# Corporate hedging and the cost of debt

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Corporate hedging and the cost of debt

by

David Christopher Alley

A thesis submitted to the graduate faculty  
in partial fulfillment of the requirements for the degree of

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This is to certify that the master's thesis of  
David Christopher Alley  
has met the thesis requirements of Iowa State University

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## 1. INTRODUCTION

As financial engineering is becoming more sophisticated and derivatives markets more liquid, firms are undertaking active risk management policies in increasing numbers (Nance, Smith and Smithson 1993). Many risk management programs include the use of financial derivatives such as options and futures to hedge exposures including cash flow variance and interest rate and foreign exchange risk. These types of hedging transactions address the risks associated with specific aspects of the firm's business. In this paper, we will develop a simple model for an internal hedge of the firm's assets, addressing risk management as a firm-wide, value creating undertaking. We will then examine the implications of such a hedging policy on the cost of debt in greater detail.

### 1.1 Modigliani-Miller

In their famous paper, Modigliani and Miller (1958) set forth a theory of capital structure in which the value of the firm is independent of the debt-equity mix of the firm. This proposition is the result of a no-arbitrage condition imposed on the prices of the equity shares of every firm in the market. Furthermore, the no-arbitrage logic of Modigliani and Miller makes unnecessary particular types of corporate hedging because investors who hold equity in a firm can always manage the risk associated with their ownership of that firm by managing their entire portfolios. If a risk averse investor feels that she has too much risk exposure by holding shares of a particular firm, she can easily remedy the situation by divesting in that firm and instead purchasing a less risky asset or portfolio of assets. These results, however, are achieved under strong assumptions of perfect capital markets, no taxes, no transaction costs, and fixed investment policy.

The Modigliani and Miller results have implications for specific types of hedging. Their work, however, does not preclude hedging certain exposures of the firm. If the hedging strategy only decreases the variance of the profit of the firm (the exact definition of profit is not important here), without increasing the expected profit, hedging will not increase the value of the firm. To be effective in increasing the value of the firm, the hedging strategy must decrease the variance of a variable in which the profit (value) of the firm is concave. It is upon this foundation that the recent hedging literature is based.

## 1.2 Concavity and Expected Firm Value

In order to judge the merits of hedging policy, we rely on Jensen's Inequality, a simple property of concave functions (see, for example, Mas-Colell, Whinston, and Green). Let  $f(X)$  be a real valued, concave function. Then Jensen's inequality states that

$$E[f(X)] \leq f[E(X)].$$

In fact, this is one way to define a concave function in two dimensions (Simon and Blume 1994). If, by hedging, we are able to decrease the variance of a variable in which firm value is concave, we are able to increase firm value. Jensen's inequality is a critical result for judging the merit of a hedging strategy. Note that the inequality reverses for a convex function. If, for example, cash flow is a concave function of asset value, then the expected cash flow from the hedged firm will be higher than the expected cash flow from the un-hedged firm if the hedge successfully decreases the variability of asset value. Thus, determining the validity of a hedging strategy involves studying the curvature of the firm's value with respect to the variable that is hedged.

### 1.3 Hedging Motivation

By relaxing the assumptions of Modigliani and Miller, many authors have suggested justifications for corporate hedging. Smith and Stultz (1985) acknowledge that, under the restrictive assumptions of Modigliani and Miller, there is no incentive for the firm to hedge. In order for hedging to increase firm value, the hedge must decrease taxes or contracting cost, or affect the firm's investment decision. All of the hedging arguments of Smith and Stultz (as well as most other authors) are dependent upon the convexity or concavity of firm value in variables such as taxes, contracting costs, and manager's wealth. This generalization follows from Jensen's inequality.

To justify hedging aimed at reducing tax liability, Smith and Stultz first assume that the amount of tax paid by the firm is a convex function of the pre-tax value of the firm. This follows from the observation that, under US tax code, the marginal tax rate is an increasing function of the pre-tax value of the firm. Under this assumption, the after-tax value of the firm is a concave function of the pre-tax value of the firm. By Jensen's inequality, the firm can increase the expected after-tax value of the firm by decreasing the variability of the pre-tax value of the firm using a hedge on firm value. This is the same type of argument that we will exploit in justifying asset value hedging.

The next issue addressed by Smith and Stultz is that of the transaction costs associated with bankruptcy as a motivation to hedge. First, assume a standard firm liability structure where the claims of equity holders are subordinate to the claims of the debt holders and there is only one class of debt. The debt holders have been promised a payment of FV (face value) at some time T in the future. The actual amount paid on that promise is dependent on the value of the firm, V, at time T. If FV is greater than the value of the firm, then the debt holders receive an amount equal to  $V - C$ , where C is the transaction cost of

bankruptcy, and the equity holders receive nothing. If the firm is solvent at time  $T$ , then the bondholders receive  $FV$  and the equity holders receive  $V - FV$ . Here, if by hedging, the firm can reduce the variability of the value of the firm at time  $T$ , then the probability that the firm will be insolvent at time  $T$  will decrease. Thus, the expected cost of bankruptcy will decrease and the value of the equity will increase. Simply, a firm that has a lower probability of bankruptcy has a higher value.

Smith and Stultz assert that, if the firm announces that it will hedge to reduce the cost of bankruptcy after the financing decision, there is little incentive for the firm to follow through on that announcement once the firm has undertaken the financing. In fact, shareholders may not want the firm to hedge at all as it will tend to redistribute wealth away from the stockholders to the bondholders in the case of bankruptcy. Thus, the promise the hedge may not be credible. Smith and Stultz, however, assert that there are circumstances under which the promise to hedge may be made credible. First, a firm that is a frequent borrower can develop a reputation for actually carrying out the promise to hedge. By hedging, the firm may also lower the cost of financial distress. The firm may be able to avoid activating covenants in their bond contracts that alter the investment decisions that the firm is able to undertake. This concept of underinvestment is a recurring theme in the hedging literature and will be used in our model as well. In my model, hedging is undertaken prior to financing, so the incentive conflict is less of an issue.

Minton and Schrand (1999) investigate the effect of cash flow variability on the firm's investment decision empirically. By regressing the level of investment on an industry adjusted measure of operating cash flow variation and two proxies for sales growth of the firm, they find that volatile cash flow is correlated with lower investment. They then run another set of regressions and find that more volatile cash flows are associated with lower



bond ratings and higher yields to maturity, suggesting that volatile cash flows do have a negative effect on the cost of capital. By hedging cash flows, the firm is able to rely less on externally generated funds and will tend to invest at a level closer to the first best outcome where all funds are costless. Later, we will formalize and simulate this effect of hedging on debt yield.

All of the motivation for hedging discussed so far do not require risk aversion, but rather rely on non-linearity in taxes and bankruptcy costs. The final motivation of hedging addressed by Smith and Stultz, however, relates to the risk aversion of firm managers. The managers' compensation is assumed to depend on the value of the firm. If the manager's wealth is a concave function of the value of the firm, then by Jensen's inequality, the manager will be strictly better off if she reduces the variability of the firm's payoffs through hedging transactions. Furthermore, if the manager is risk averse, even if wealth is not concave in firm value, there may be an incentive for the manager to hedge. As long as there is a positive relationship (linear, concave, or convex) between firm value and the manager's wealth the manager can still maximize expected utility by hedging because the manager's utility is a concave function of firm value under risk aversion. Stultz (1984) developed a continuous time model specifically for hedging exchange rate risk under similar assumptions of management compensation in which he found that, with risk averse managers, the firm will hedge. DeMarzo and Duffie (1995) take this analysis further and examine the role of asymmetric information between managers and shareholders and the incentive to hedge. They find that, based on the accounting standards used, managers who are not forced to fully disclose hedging activities have an incentive to fully hedge the firm's accounting profits. Those who must fully disclose hedging accounts may not hedge at all.

Many arguments have been developed to support the practice of corporate risk management. In this paper, we will show that, if the value of the firm is represented as an uncertain cash flow dependent upon the level of investment and the cost of external financing of that investment, then the firm value can be increased through hedging transactions.

#### **1.4 Froot, Scharfstein and Stein's Cash Flow Model**

One straight forward cash flow model by Froot, Scharfstein, and Stein (1993) (FSS henceforth) demonstrated that, with costly external financing convex in the level of external financing and a concave production function, firm value can be increased by undertaking hedging transactions (for a brief description of the model, see Appendix A). The effectiveness of the hedge depends on the amount of curvature of the firm's production and cost of debt functions. This is intuitively appealing because as the amount of external financing reaches very high levels, investors will demand a high return on any debt instruments that they purchase from the firm and, thus, the cost of taking on even more debt increases. In the extreme case, external financing becomes infinitely expensive as capital markets become inaccessible to debt-laden firms.

The FSS model is based on the assumption that market imperfections make external financing more expensive than internal financing. If cash flows are variable, then the firm has two choices when financing investment. They can either vary the amount of investment, or they can vary the amount of externally generated funds. FSS find that the firm will choose a level of investment that is closer to the optimal amount of investment if the cost of financing is decreased through hedging. The model is limited, however, because it only considers financing cost as a function of the level of external financing. This model addresses only the scale of external financing without regard to the amount of assets that the

firm has to repay the sources of external finance. We will derive a result similar to that of the FSS model using a cost function that is a function of not only the level of external finance, but also the value of the firm's assets. The model that we will develop will include the underinvestment problem, but will also show how the cost of debt changes with hedging. The theoretical model that follows helps to explain the empirical results of authors such as Minton and Schrand (1999).

## 2. THE MODEL

### 2.1 Operating Cash Flow

Following FSS, let  $w$  be the time zero value of the liquid assets, and  $V$  be the value of other productive assets net of existing liabilities. Thus,  $V$  is the net worth of the firm excluding liquid assets. Both cash holding and asset value are random variables which are independent. Therefore, at time zero, total value of the firm's assets is given by:

$$A = w + V$$

Also at time zero, the firm chooses the level of investment,  $I$ . This investment includes all cash outflows spent to support sales and capital expenditures. The firm earns a return on this investment as well as a return on real assets already held by the firm at time zero. The one period value of a given level of investment and time zero assets with a zero discount rate is given by:

$$f(I) + g(V) - I$$

where  $f(I)$  is the expected level of cash inflow (a money metric production function) given the level of cash outflow,  $I$ .  $g(V)$  is the expected cash inflow on existing assets.

Combining the two pieces of the production function into a single function,  $h(I, V)$ , does not change the results derived below. Also, assume that  $f_I(I) > 0$  and  $f_{II}(I) < 0$ . Then, the time-zero present discounted value of a given level of investment in a risk neutral world is given by:

$$[f(I) + g(V)]\exp(-rT) - I$$

where  $r$  is the continuous risk free interest rate and  $T$  is the amount of time in years between time 0 and the time when the cash inflows are realized. We assume that all cash inflows occur discretely at time one. The risk neutral assumption alleviates the need to calculate an

expected return on equity to discount cash flows because, in the risk neutral world, the expected return on any investment is simply the risk free rate.

## 2.2 Financing Cost

The firm has two sources of funding for the optimal level of investment,  $I^*$ . Funds can be generated internally from liquid assets or from external sources. All external financing, represented by  $e$ , will be obtained through the sale of zero-coupon bonds. As the value of the firm's assets is stochastic, the firm has some positive probability of becoming insolvent and defaulting on the promise to pay the face value of the bond at maturity. Thus, these bonds are risky to the bondholder and will be sold for some amount less than the present value of face value when issued by the firm. We assume that the price of the bonds,  $P^b$ , is less than the price paid for a risk free bond with identical terms and maturity as the risky issue. A model of default will be discussed later. The analysis could be extended to include coupon bonds and bonds of varying maturities, but for the sake of clarity, we will address only the zero coupon case where the maturity date of the bond matches the length of the production cycle so that the timing of cash inflows matches the timing of the repayment of the bonds. Furthermore, the firm cannot raise additional external funds by issuing equity.

Because the bonds are sold at a price below that of risk free bonds, there is a deadweight cost associated with obtaining funds from external sources. Hence, the firm will choose to fund investment first by using internally generated funds and then turning to external sources to raise the difference between  $w$  (internally generated funds) and  $I^*$  (optimal level of investment). Therefore, once  $I^*$  is chosen,  $e$  is given by:

$$e = I^* - w,$$

where  $w$  is a random variable and  $I^*$  is the solution to a profit maximization problem presented below. This does not necessarily imply that the firm uses all of its cash to finance investment. Obviously, most firms hold a certain amount of cash at all times. Here, assume that the portion of cash that the firm withholds from investment financing will be a portion of  $V$  instead of  $w$ .

### 2.3 The Cost Function

To explore the motivation to hedge, it is useful to first develop some general characteristics of the cost of external financing. If the firm were only to sell one bond to finance investment, then the present value of the cost of financing is given by the following:

$$C = FV \exp(-rT) - P^b$$

This cost is the time-zero present value of cash outflow from external financing less the cash inflow from external financing. Now, suppose that instead of selling one bond, the firm obtains funds externally by issuing \$1 face value bonds. The firm sells  $n$  such bonds. Then, the total face value of the bonds is  $n$  and the net present cost of external finance per dollar of face value is given by:

$$C = n[\exp(-rT) - P^b]$$

where  $P^b$  is now the time-zero price of the \$1 face value bond. To simplify, note that, in the risk neutral world, the discounted value of a dollar received at time  $T$  is equivalent to the price of the risk free bond at time zero,  $P^f$ . Then the above equation can be rewritten as follows:

$$C = n(P^f - P^b)$$

In this expression, the total cost of external financing is the difference between the price of a dollar lent and the price of a dollar borrowed multiplied by the number of bonds sold.

Assume that the firm has perfect foresight of the value of  $P^b$  in the short time interval prior to the sale of the bonds. The firm will then sell a number of bonds,  $n$ , such that

$$n = \frac{e}{P^b}.$$

Substituting into the cost function, the total cost of external financing can be written as:

$$C = e \left( \frac{P^f}{P^b} - 1 \right)$$

In a departure from the FSS model, the cost of external financing is a function of both the amount of external financing,  $e$ , and the total value of assets at time zero,  $A$  because it is assumed that  $P^b$  is a function of  $A$  and  $e$  which will be denoted  $P^b(A, e)$ . This coincides with the Merton-Black-Scholes bond pricing model that will be used later, but we can obtain certain general properties of the cost function as a result of this bond price structure. These general properties will hold under other bond pricing models as well.

To simplify notation, let

$$\frac{P^f}{P^b(A, e)} - 1 = y(A, e),$$

a measure of real bond yield. Then, since  $e$  is non-negative and noting that

$$\frac{de}{dA} = \frac{dl}{dA} \geq 0^1,$$

we can derive the general properties of  $C(P^b, e) = C(A, e) = ey(A, e)$  by deriving the curvature properties of  $y(A, e)$ . Intuitively, as the value of assets increases for a given level of debt financing, the firm will be less likely to default on those debt commitments. Thus, investors will be willing to pay more for a bond issued by a firm with a higher asset value.

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<sup>1</sup> This result comes from the identity  $e = I - w$ . Then  $de/dA = dl/dA - dw/dA = dl/dA$ . We then assume that an increase in asset value will not have a negative affect on investment.

Therefore,  $y_A(A, e) < 0$ . Next, it is useful to think of the cost of financing in the limit as  $A \rightarrow 0$  and  $A \rightarrow \infty$ . As the value of the assets of the firm tends towards zero, investors will not be willing to purchase the debt issued by the firm, as there are not sufficient assets to keep the firm solvent. In that case, the price of the bonds tends towards zero. As we can see from the definition of the bond yield,

$$\lim_{A \rightarrow 0} y(A, e) = \lim_{A \rightarrow 0} \left( \frac{P^f}{P^b(A, e)} - 1 \right) = \infty$$

If the value of assets gets very small at the time of bond issue, the bond yield gets large because the firm is not able to access debt markets. As the value of the assets tends towards infinity, the probability of default becomes small and investors will only demand a return that is equal to the risk free rate of return. Again, from the definition of bond yield, we have the following:

$$\lim_{A \rightarrow \infty} y(A, e) = \lim_{A \rightarrow \infty} \left( \frac{P^f}{P^b(A, e)} - 1 \right) = \frac{P^f}{P^f} - 1 = 0$$

Thus, externally generated funds have the same cost as internally generated funds when the firm has a very large asset value.

Next, consider the level of debt,  $e$  and the yield function. As the level of debt increases, *ceteris paribus*, there is a greater chance that the firm will default. This will push the price of the bonds down. Therefore,  $y_e(A, e) > 0$ . As  $e$  tends towards zero, the price of the bonds will tend toward the price of the risk free bond. Thus,

$$\lim_{e \rightarrow 0} y(A, e) = \lim_{e \rightarrow 0} \left( \frac{P^f}{P^b(A, e)} - 1 \right) = \frac{P^f}{P^f} - 1 = 0$$



External financing is costless for small levels of debt assuming that  $A$  is sufficiently large. Finally, when the level of debt becomes very high for a given level of assets value, investors will not be willing to pay a very high price for that debt, and in the limit, will not be willing to pay anything for the debt. Thus,

$$\lim_{e \rightarrow \infty} y(A, e) = \lim_{e \rightarrow \infty} \left( \frac{P^f}{P^b(A, e)} - 1 \right) = \infty$$

For very high levels of debt, the cost of debt tends towards infinity and the debt market becomes inaccessible to the firm.

In summary, these characteristics of  $y(A, e)$  imply the following:

- |                       |                       |
|-----------------------|-----------------------|
| 1) $C_A(A, e) < 0$    | 3) $C_e(A, e) > 0$    |
| 2) $C_{AA}(A, e) > 0$ | 4) $C_{ee}(A, e) > 0$ |

Together, these characteristics imply that the cost function is convex in both  $A$  and  $e$ .

## 2.4 Profit Maximization

Following FSS, we maximize profit by working backwards. First, we choose the cash flow maximizing level of investment at time one (discounted back to time zero):

$$\max_I \{ [f(I) + g(V)]P^f - I - C(A, e) \}$$

The first order condition for this maximization problem is:

$$\left[ f_I(I) + g_V(V) \frac{dV}{dI} \right] P^f - 1 - C_A(A, e) \frac{dA}{dI} - C_e(A, e) \frac{de}{dI} = 0$$

Noting that

$$\frac{dV}{dI}, \frac{dA}{dI} = 0 \text{ and that } \frac{de}{dI} = 1 - \frac{dw}{dI} = 1,$$

the first order condition can be written as:

$$f_I(I)P^f - 1 - C_e(A, e) = 0$$

First, note that with when financing only with internally generated funds or costless external financing, the firm will set

$$f_I(I) = \frac{1}{P^f}.$$

The firm will increase investment until the return on investment is equal to the gross return on a risk free bond. The cost of financing is the opportunity cost of those funds which we assume to be the risk free return. Thus, the firm will use internally generated funds to finance investment up to the point where the expected return on the investment is equal to the risk free return. This is a standard marginal product outcome where the firm continues to invest until they don't realize a positive net return by spending more.

Turning to the case where external financing is costly, the firm will continue to invest until the marginal value product of investment is equal to the return on a bond used to finance that investment. Additionally, with costly external financing, the firm will invest at a level that is less than the first best outcome of free external financing. Since  $C_e(A, e) > 0$ , the firm continues to invest until

$$f_I(I) = \frac{1 + C_e(A, e)}{P^f} > \frac{1}{P^f}.$$

This illustrates the underinvestment problem that occurs with costly debt financing. If, by hedging, the firm can reduce  $C_e(A, e)$ , then the firm is able to invest more and realize a higher gross return.

A necessary condition for this solution to be a maximum is that the second derivative of cash flow with respect to investment must be negative:

$$\begin{aligned} f_{II}(I)P^f - C_{ee}(A, e)\frac{de}{dI} - C_{eA}(A, e)\frac{dA}{dI} < 0 \\ \Rightarrow f_{II}(I) < \frac{C_{ee}(A, e)}{P^f} \end{aligned}$$

This condition always holds under the assumptions that  $f(I)$  is concave and  $C(A, e)$  is convex in  $e$ .

## 2.5 Justification for Hedging: The Non-Linearity of Cash Flows

In the previous section, we derived profit maximizing conditions for the optimal choice of investment. What remains to be found is under what conditions the firm can increase firm value by hedging. As before, assume that  $f_I(I) > 0$  and  $f_{II}(I) < 0$ . Furthermore, to simplify the following argument, we will use a generalized function for the cost of external financing. Again, let  $C(A, e)$  represent the time 0 total cost of external financing. The  $C(A, e)$  function has a negative first derivative with respect to the firm value and is convex in firm's value.

The time zero present value of the cash flows of the firm is given by<sup>2</sup>:

$$CF_0(A, I, e) = [f(I) + g(V)]P^f - I - C(A, e)$$

---

<sup>2</sup> Throughout this paper, we denote the discounted cash flow between time zero and time one as  $CF_0$ .

To justify hedging, it is necessary to show that this function is concave in  $w$ , the only variable that we are able to change by hedging (viewing the hedge as a decreasing total asset value yields the same results). Therefore, it is sufficient to show that:

1.  $CF_w(A, I, e) > 0$
2.  $CF_{ww}(A, I, e) < 0$

First, consider the first derivative of cash flow with respect to  $w$ :

$$CF_w(A, I, e) = \left[ f_l(I) \frac{dI}{dw} + g_v \frac{dV}{dw} \right] P^f - \frac{dI}{dw} - C_A(A, e) \frac{dA}{dw} - C_e(A, e) \frac{de}{dw}$$

Using

$$\frac{dV}{dw} = 0, \frac{dA}{dw} = 1$$

and, by profit maximization,  $f_l(I)P^f - 1 - C_e(A, e) = 0$ , the first derivative with respect to  $w$  can be rewritten as:

$$\begin{aligned} & \frac{dI}{dw} \left[ f_l(I)P^f - 1 - C_e(A, e) \right] + C_e(A, e) - C_A(A, e) \\ & = C_e(A, e) - C_A(A, e) \end{aligned}$$

It then follows that the second derivative with respect to  $w$  is given by:

$$\begin{aligned} CF_{ww}(A, I, e) &= C_{ee}(A, e) \frac{de}{dw} + C_{eA}(A, e) \frac{dA}{dw} - C_{AA}(A, e) \frac{dA}{dw} - C_{eA}(A, e) \frac{de}{dw} \\ &= \left( \frac{dI}{dw} - 1 \right) \left[ C_{ee}(A, e) - C_{eA}(A, e) \right] + C_{eA}(A, e) - C_{AA}(A, e) \end{aligned}$$

Before further development of this expression, it is useful to examine some simplifying cases of  $dI/dw$  to extract some intuition. First, consider the case where, for a one unit change in liquid assets, the firm spends two additional units, or  $dI/dw = 2$ . Then,

the cash flow function is concave if  $C_{ee}(A, e) < C_{AA}(A, e)$ . To further simplify this result, it is first helpful to notice that  $C_e(A, e) = MC(e)$  is the marginal cost of external finance.

Similarly,  $C_A(A, e) = -MB(A)$  is the negative of the marginal benefit of additional real assets or, more importantly, liquid assets (increased assets lowers borrowing cost). Then, the necessary condition for concavity can be rewritten as

$$\frac{\partial MC(e)}{\partial e} < -\frac{\partial MB(A)}{\partial A}.$$

When hedging increases  $w$ ,  $A$  increases as well. Hedging then increases the marginal benefit of external financing,  $MB(A)$ , because of the convexity of the cost function in  $A$ . The other effect of an increase in  $w$  is that  $e$  increases by one unit. This will increase the marginal cost of debt financing,  $MC(e)$ , because of the convexity of the cost function in  $e$ . Therefore, the condition that  $C_{ee}(A, e) < C_{AA}(A, e)$  tells us that hedging is value creating if the hedge will increase the marginal benefit of the increase in cash more than it will increase the average cost as a result of the additional external financing that results from the increase in cash.

Informally, this condition implies some characteristics of the hedging decision. First, hedging is not justified when  $A \rightarrow \infty$  because of the following characteristics of the cost function:

$$\lim_{A \rightarrow 0} C_A = 0 \text{ and } \lim_{A \rightarrow 0} C_{AA} = 0$$

On the other hand, hedging will also not be justified for the range of the cost function that is very convex in  $e$ . This restriction, however, is not as clear as the restriction on the cost function with respect to  $A$ . If the cost function is very concave in both  $e$  and  $A$ , then it is less clear whether or not to hedge.

Next, we consider the case where investment is fixed, or  $dl/dw = 0$ . Cash flow is concave if:

$$C_{eA}(A, e) < \frac{C_{AA}(A, e) + C_{ee}(A, e)}{2}$$

Under the convexity assumptions imposed on the cost function, the right-hand-side of this equation is always positive. In certain cases, such as when  $A$  is very large and  $e$  is very small, the right-hand-side will tend towards zero because the cost function is not convex in the limit in  $A$  and  $e$ . Furthermore, the cross-partial derivative on the left-hand-side of the inequality will always be negative except in the limit when  $A \rightarrow \infty$  or  $e \rightarrow 0$ . That is, the marginal effect of an increase in  $A$  on the marginal cost of debt with respect to  $e$  will be negative. More intuitively, changes in the amount borrowed have a smaller effect on the average cost of debt for larger asset values. Considering all of these characteristics, the inequality is always true, except when  $A \rightarrow \infty$  and  $e \rightarrow 0$ . Therefore, except for cases of extreme asset values and debt level, hedging will increase the expected cash flow of the firm.

Next, consider the case when  $dl/dw = 1$ . Here, additional cash is invested when liquid assets increase, but no additional funds are borrowed. Cash flow is concave in  $w$  if

$$C_{eA}(A, e) < C_{AA}(A, e)$$

In contrast to the previous cases,  $C_{ee}(A, e)$  is no longer a concern as changes in  $w$  will not change  $e$ . Thus, there is no change in marginal cost of  $e$  with respect to  $e$ . There is, however, a change in the average cost of  $e$  with respect to  $A$ . In this case, the change in the average cost of  $e$  with respect to  $A$  must be less than the change in marginal benefit with respect to  $A$  when  $A$  changes. From the earlier discussion of the  $dl/dw = 2$  case, this condition will always hold except when  $e$  is very small or  $A$  is very large.

Of course, it cannot be assumed that any of these simple cases is necessarily true, so we need to derive an expression for  $dI/dw$ . First, recall the first order condition that  $f_l(I)P^f - 1 - C_e(A, e) = 0$ . Taking the total differential, we have the following:

$$f_{ll}dIP^f - C_{ee} \frac{de}{dw} dw - C_{eA} \frac{dA}{dw} dw = 0$$

Substituting

$$\frac{dA}{dw} = 1 \text{ and } \frac{de}{dw} = \frac{dI}{dw} - 1,$$

$dI/dw$  is written as follows:

$$\frac{dI}{dw} = \frac{C_{eA} - C_{ee}}{f_{ll}P^f - C_{ee}}$$

Then, rewrite the second derivative of cash flow as (dropping functional notation):

$$\begin{aligned} CF_{ww} &= \left[ \frac{C_{eA} - C_{ee}}{f_{ll}P^f - C_{ee}} - 1 \right] [C_{ee} - C_{eA}] + C_{eA} - C_{AA} \\ &= \left[ \frac{C_{eA} - f_{ll}P^f}{f_{ll}P^f - C_{ee}} \right] [C_{ee} - C_{eA}] + C_{eA} - C_{AA} \end{aligned}$$

Therefore, cash flow is concave if:

$$f_{ll}P^f < \frac{(C_{eA})^2 - C_{AA}C_{ee}}{2C_{eA} - C_{AA} - C_{ee}}$$

By signing each element of this expression, it can be shown that the inequality is always true if  $(C_{eA})^2 \leq C_{AA}C_{ee}$  and the production function is concave. Therefore, hedging may not be necessary in the cases where the cost function is not very convex in assets and/or the level of

debt. Again, this will occur when the value of the firm's assets or the level of debt is very large or very small. Furthermore, if it is assumed that  $dI/dw \geq 0$ , then

$$\frac{C_{eA} - C_{ee}}{f_{II}P^f - C_{ee}} \geq 0$$

and we can then show that hedging is always justified if  $C_{eA} \leq C_{AA}$ . Thus, the necessary condition collapses to the  $dI/dw = 1$  case. Notice also that, in a departure from the FSS model, there may be cases when concavity of the production function is not necessary to justify hedging. A sufficiently convex cost function is enough to motivate hedging.

## 2.6 Hedging and Share Value

While we will not empirically examine the effect of hedging on share price, we can show that share price will increase for two reasons. We will appeal to a simple discounted cash flow model of firm valuation (see, for example, Ross, Westerfield, and Jaffe (2002) or Brealy and Myers (2000)). In this model, the value of the equity of the firm is equal to the discounted value of all future cash flows to equity. The discount factor in this model is the expected return to equity.

In the simplest version of the discounted cash flow model, the firm has constant expected cash flow and expected return on equity forever. In this case, the value of the firm's equity is given by:

$$Value = \sum_{t=1}^{\infty} \frac{E(CF)}{(1+r_E)^t} = \frac{E(CF)}{r_E}$$

Note that the definition of cash flow developed previously is cash flow to equity because it is net of debt service.



If, instead, we assume the firm only exists for one period (as in our formal model), the value of the firm is given by:

$$Value = \frac{E(CF)}{1 + r_E}$$

In this simple valuation model, the value of the firm will increase with hedging for two reasons. First, for the one period model, we showed that hedging will increase cash flow. Thus, CF increases. Furthermore, the expected return on equity decreases. This follows from the capital asset pricing model (CAPM).

According to the CAPM, the expected return on equity is given by the following:

$$r_E = r_f + \beta(r_M - r_f)$$

where  $r_M$  is the expected return on a portfolio of all market assets and

$$\beta = \frac{Cov(r_E, r_M)}{\sigma_M^2}$$

In practice, the market portfolio is often a broad equity index. With hedging, the value of  $\beta$  will be at most equal to the value of  $\beta$  without hedging except in the unlikely case when the return on the hedging index and the return on equity are both negatively correlated with the market portfolio<sup>3</sup>. This follows from the definition of covariance:

$$Cov(r_E, r_M) = \rho_{r_E r_M} \sigma_{r_E} \sigma_{r_M}$$

First, we assume that the standard deviation of the market portfolio ( $\sigma_{r_M}$ ) is constant. Then, because the value of the firm's assets is hedged, the volatility of the return on those assets will decrease. Hedging will also act to decrease the correlation between the return on the

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<sup>3</sup> One commonly used proxy for the market portfolio is the S&P 500 index. The return on the NASDAQ 100 index and the return on Intel stock and assets (the hedging index and the firm hedged, respectively) are both highly positively correlated with the S&P 500 return.

market portfolio and the return on the equity of the firm. Put simply, a decrease in market risk decreases the expected return on equity, thus increasing the current value of the firm's equity.

### 3. SIMULATION METHODS

To examine the distributional implications of cash flow of this model, we construct a simulation of the cost-of-debt portion of cash flow. While it might be more informative to simulate all of cash flow of the firm, this presents many estimation problems which would make the simulation unnecessarily complicated. Thus, the simulation will tend to understate the benefits of hedging, but will demonstrate that the reduction in financing cost alone should motivate the firm to hedge. Also, as was shown in the last section, the decision to hedge is driven by the cost function.

To simulate the model, we use the stochastic nature of the asset values to generate a large number (5000) of possible outcomes. In this case, we wish to simulate different realizations of the cost of debt of the firm both when the firm hedges and when it does not. We then average over all of those realizations to obtain an estimate of the expected cost of debt. It is important to use a high number of sample realizations in order to have convergence of the sample mean to the true mean. Here, we appeal to the weak law of large numbers that, as the size of the sample tends towards infinity, the sample mean approaches the true mean (see, for example, Miller, Miller, and Freund 1998). The “true mean” is not meant to represent the true mean of bond yields in the real world, but rather the true mean given the distributional assumptions of the parameters of the model.

#### 3.1 Simulating the Complete Model

Recall that total discounted one period cash flow is given by the following:

$$CF_0(A, I, e) = [f(I) + g(V)]P^f - I - ey(A, e)$$

The production function cannot be directly estimated, but we could model  $f(I) + f(V)$  as operating cash inflow of the firm by estimating a stochastic statistical model from historic

accounting operational cash flow as a random variable correlated to investment in the previous period. Capturing the non-linearity of the production functions would be very difficult using this method.

Another obstacle in simulating the entire model is that it requires simulation of liquid asset and real asset values separately as the investment decision is based only on the amount of liquid assets available. This requires another estimated correlation relationship between the two types of assets. A related issue is the necessity of estimating an investment rule. One possible way to estimate the investment rule is to regress historical investment on the level of liquid and real assets. This method would result in an under-investment simulation, but it has little grounding in the actual model in that no optimization is taking place. Furthermore, this method would only be valid for a linear production function, violating one of the basic assumptions of the model.

### 3.2 Hedging the Cost of Debt

Because of the shortcomings in simulating the entire model, we will focus only on the cost of debt. This is similar to the case discussed earlier when the firm's investment decision is fixed, or  $dl/dw = 0$ . From the earlier discussion of conditions under which hedging is justified, we expect that hedging will always be value creating with a fixed investment policy with the possible exception of extreme asset values and amounts of borrowing. We will only be concerned with the distribution of  $ey(A, e)$  with and without hedging. Furthermore, since that  $e$  is a scaling factor, we can focus on the "discounted" bond yield:

$$y(A, e) = \frac{P^f}{P^b(A, e)} - 1.$$

The simulation spans three discrete points in time. In the simulation, the length of the interval between integer time indexes is one year. The time periods and the corresponding events are as follow:

$t = -1$ : The firm places the hedge by purchasing a put option on an equity index that is correlated with the value of the firm's assets. Assume that the put option is fairly priced. That is, the price of the put is equal to the discounted expected payout of the option given the distributional assumptions of the underlying asset. The underlying asset here is the value of the NASDAQ 100 index, an index on which options are actively traded. The fair pricing assumption implies that, on average, the discounted payout of the option is equal to the price of the option and the net cost of the hedge is zero. Also at time -1, we determine the value of the firm's assets ( $A_{-1}$ ) and the volatility of the firm's assets ( $\sigma_A$ ).

$t = 0$ : A value of the firm's assets is realized following the asset value diffusion model discussed below. The put option that was purchased at  $t = -1$  pays off and is added to the value of the firm's assets. The firm borrows an amount  $e$  by issuing zero coupon bonds that mature in one year.

$t = 1$ : The face value of the funds borrowed at time zero are repaid to the bondholders. There is no need to simulate realizations of asset values at this time for our purposes. The amount of time that passes between  $t = 0$  and  $t = 1$  is needed to price the bonds that are issued at  $t = 0$ .

The simulation is primarily a matter of estimating bond prices for different realizations of asset values with and without the hedge. We use a Merton-Black-Scholes model of debt pricing, but this simulation could be done with a number of different pricing models. The exact form of the pricing model is not the driving factor for the decision to hedge, but rather the curvature of the bond yield with respect to asset value and level of debt.

### **3.3 Bond Pricing Models**

There are countless bond pricing models that depend on a number of different factors. Here, we will follow the arguments of Black, Scholes (1973) and Merton (1974) to develop a simple theoretical framework that represents a bond as an option. The hedging model could be adapted to accommodate a variety of bond pricing models some of which may be more empirically tractable. Bond pricing models can generally be classified in three main categories (Nandi 1998). The first category of models is credited to Black, Scholes, and Merton (henceforth BSM). As described below, this model uses the BSM option pricing model to price the bond which is inherently an option on the value of the firm's assets. A closed-form equation is then used to value the option. While theoretically appealing, this model is often criticized, however, because its implementation requires the value of the firm's assets as an input which may be difficult to ascertain in that simple accounting measures of the firm's asset value may be an inaccurate assessment of the true value of assets. As discussed later, work by Jones, Mason, and Rosenfeld (1984) and more recently by Hull, Nelken, and White (2003) offers a market driven solution to this asset valuation issue. Furthermore, the BSM model only allows for default on the bond at maturity. Also, the BSM framework is only applicable to the zero-coupon bond, but many others such as Vasicek have extended the model to include coupon bonds. Finally, the BSM model is

attacked because it assumes that interest rates are fixed (Longstaff and Schwartz, 1995). Despite its shortcomings, the BSM model is useful here, as it implicitly includes the probability of bankruptcy and gives the cost of external financing an explicit function. Furthermore, the magnitude of bond yields produced by the model is not as important as the spread between the hedged bond yield and the un-hedged bond yield. Crosbie and Bohn (2002) find that the BSM model is useful in ranking the yields of different firms. Therefore, it is reasonable to use the BSM model, as we are interested in the spread between the hedged and unhedged bond yields.

The next category of model is the structural models. These models weaken many of the assumptions of the BSM model. (Actually, the BSM model is a simple structural model.) First, they are more realistic with respect to timing of default. While the BSM model only allows for default at maturity, structural models allow for default at any time between issue and maturity when the firm is insolvent. These models further assume that the bondholders receive some fraction of the face value of the debt in the event of default. This fraction, called the recovery ratio is known at the issue date. These models are also based on the value of the firm's asset, so may be difficult to estimate, just as the BSM model.

The final group of models is the reduced form models. These models have the advantage over structural models and the BSM model that a value of firm assets is not required. These models assume that the probability of default is driven by an exogenous random variable. Furthermore, using an expected recovery ratio given that default has occurred, the bond can be priced by its expected payoff. Default probabilities can be estimated using historical default probabilities based on a firm's bond or credit ratings by Fitch, Standard and Poor's, or Moody's (Delianedis and Geske 1999). One such model by Longstaff and Schwartz (1995) incorporates both a default probability model and an interest

rate risk model which results in more empirically supportable results than the BSM model. While these models would generate the best empirical results in terms of the magnitude of the bond yield, they would be problematic in the hedging simulation. The time -1 credit rating of the firm is easily observable. It is not a simple matter, however, to simulate the time zero bond rating. First, bond ratings are not calculated using any single metric of debt serviceability. Rating agencies base the firm's credit rating on a number of factors including debt to value and cash flow to debt service, but the rating is also based on less tangible measures such as the firm's future business prospects and the competitive climate of the industry. Therefore, it is difficult to formulate a rule to simulate ratings migration. Furthermore, in practice, there may be a lag between the deterioration of the firm's financial stability and a change in rating. Finally, a particular rating may cover a range of debt to value levels, so there is significant non-continuity in the yield curve.

### **3.4 The Black-Scholes-Merton Model**

In the Black-Scholes-Merton model, a bond is viewed as a long position in cash and a short put option<sup>4</sup> (see also Giesecke 2003). First, suppose that the firm issues zero-coupon bonds. The zero-coupon bonds are sold in an auction system, presumably at a discount from the face value of the bonds (assume a positive discount rate equal to the risk free rate for potential bondholders). If the bonds were not sold at a discount, then they pay a negative interest rate. This is a situation that we will not address here. Furthermore, there exists a default free zero-coupon bond with identical maturity to the bond issued by the firm. The continuous return on the default free bond is given by  $r$ . In practice, the default free bond is a

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<sup>4</sup> We could view the bond as this same long position and short put option and use an option pricing method other than the Black-Scholes-Merton method.



US government issue such as a treasury bill for short maturities or a treasury bond for long maturities.

The purchasers of the corporate bonds are promised a single payment equal to the total face value of the bonds at maturity. The only other claimants on the firm's assets are the equity holders. The equity holders' claims on the assets of the firm are subordinate to the claims of the debt holders. That is, in the case of bankruptcy, bondholders are entitled to the assets of the firm in an amount equal to the face value of the bonds that they hold. After the bondholders have been paid, equity holders are entitled to the assets that remain. More formally, if, at maturity of the bonds, the value of the firm's assets per bond,  $a_T = A_T/n$ , is greater than the face value of each bond (\$1), then the debt holders will be paid \$1 per bond. The equity holders are then left with assets with a total value of  $n(1 - a_T)$ . If, on the other hand, the face value of the debt is greater than the value of the assets, the firm will be in bankruptcy. Bondholders will receive  $a_T$  per bond at maturity. Some models depart from this strict definition of bankruptcy (Crosbie and Bohn 2002). Such models are based on the empirical observation that few firms actually file bankruptcy when asset value falls below the level of liabilities because of the long-term nature of some of the debt. Instead, there is a "default point" at which bankruptcy actually occurs. The default point is characterized by an asset value that lies between short-term liabilities and total liabilities. For the BSM model used here, we only have one bond issue, so there is no distinction between long-term and short-term liabilities generated by financing activities. All debt matures simultaneously.

As mentioned above, there are two possible outcomes in this model at time T. In the first outcome, where  $a_T \geq 1$ , the firm will pay the bondholders a total amount n. In the other outcome, where  $a_T < 1$ , the firm does not have enough assets to pay back the face value of

the bonds. The bondholders receive the entire value of the firm's assets. Thus, the value of a bond (to the bond holder) at maturity is given by:

$$\min(FV, a_T) = \min(1, a_T) = 1 - \max(1 - a_T, 0)$$

which is simply the face value of the bond less the value of a put option on the value of the firm's assets with strike price equal to the face value of the bond. *Figure A* is a representation of the payoff of this option. If the value of the firm's assets is zero, then the bond is worthless. As the value of the assets increases, but is still less than the face value of the bond, there is a one-to-one relationship between the value of the firm's assets and the value of the bond. If the value of the firm's assets at time T is greater than the face value of the bond, then the value of the bond is equal to the face value of the bond.

### 3.5 Valuing the Contingent Claim

As is apparent from the position diagram in *Figure A*, it is a simple exercise to determine the value of the bond at maturity. The value of the firm's assets is known with certainty at time T, so the payoff of the bond is known with certainty. Valuing the option prior to expiration is a more difficult and widely addressed task. For this European style option (or an American option for that matter), it is necessary to determine the present value of the expiration payoff. This, of course, is made difficult by the uncertainty of that payoff. At time t we do not know how much the option will pay at time T or if it will payoff at all.

There are a number of factors that affect the price of an option at time t including the current level of the underlying variable ( $a_t$ ), the option strike price (FV, here assumed to be \$1), the time to expiration (T-t), the volatility of the underlying variable, and the risk free interest rate (Hull 2000). All of these factors affect the distribution of the payoff on the option, and thus, the price of the option. The value of the put option will increase when  $a_t$

decreases, when FV increases, when the volatility of A increases, and when the risk-free rate decreases. In turn, these factors will all determine the price of the bond at time zero.

The BSM model results in an explicit function to value the option. Since a bond is a combination of a long cash position and a short put option, put option form of the BSM model can be applied directly to the bond. With the strike price equal to the face value of the bond (\$1), the price of the bond is given by the following:

$$P^b = FVe^{-rT} - [FVe^{-rT} N(-d_2) - A \times N(-d_1)]$$

where

$$d_1 = \frac{\ln\left(\frac{A}{FV}\right) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma_A \sqrt{T}}$$

$$d_2 = d_1 - \sigma_A \sqrt{T}$$

and  $r$  is the continuous risk free interest rate,  $\sigma_A$  is the volatility of the value of the assets of the firm,  $T$  is the time in years from time zero to maturity of the debt, and  $N(\bullet)$  is the cumulative normal distribution with mean zero and variance one. We can use this pricing model to study how different levels of debt (FV) affect the distribution of debt cost with and without hedging.

For the practical matter of simulation using Intel Corporation data, we ran the simulation using the following estimates for the parameters in the bond pricing model (all estimates are on a per share basis):

1.  $r$ : The continuous return on a one United State Treasury Bill as reported as a discrete return in the Wall Street Journal.
2.  $A$ : The per share value of the firms assets at time zero is determined by a method described later using the volatility and current price of the firm's common stock.

3.  $\sigma_A$ : The volatility of firms assets is found simultaneously with asset value.
4. T: The time horizon is arbitrarily set at one year.
5. FV: It would be very simple to run the simulation using a fixed face value of the bonds for each realization of asset value at time zero. Doing that, however, result in variation in the amount of money borrowed ( $P^b$ ) for every different realization of asset value at time zero. Therefore, we chose to fix the amount borrowed and solve for the face value that is implied by that fixed amount of borrowing for each realization in the simulation. Since the BSM bond pricing model cannot be solved for FV, we have to use an iterative search technique to find the face value of the bond at which the price of the bond is equal to  $e$ . In our simulation, this was done using Solver in Microsoft Excel.

Values of each parameter are shown in *Table A*.

### 3.6 Asset Diffusion

To simulate the distribution of financing cost, we construct a simulation based on a lognormal diffusion of the value of assets. As we will show, a lognormal distribution of asset values follows from normally distributed asset returns. This method is made necessary by the use of the Black-Scholes-Merton model of debt valuation. The normality assumption has its roots in literature that predates the work by Black, Scholes, and Merton, and has been a point of contention since its inception. For example, Fama (1965) found that daily stock returns have a distribution that is more kurtotic than a normal. His work suggested that stock returns follow a Paretion distribution. Other authors, however, have found that monthly returns are normally distributed (Blattberg and Gonedes 1974).

### 3.6.1 Geometric Brownian Motion and Lognormal Prices

The BSM results are built upon the assumption of geometric Brownian motion of asset values (Hull 2000 and Durrett 1996). That is, the change in the value of assets is given by:

$$dA = \mu_A A dt + \sigma_A A dz$$

where  $\mu_A$  is the drift rate of the asset value and  $dz$  is a Weiner process. Thus, in discrete terms,  $\Delta z = \varepsilon \sqrt{\Delta t}$  where  $\varepsilon$  is a random draw from a standard normal distribution. It then follows from Ito's Lemma (for derivation, see Appendix B) that

$$d \ln A = \left( \mu_A - \frac{\sigma_A^2}{2} \right) dt + \sigma_A dz$$

Therefore,

$$\ln \frac{A_T}{A_0} \sim \phi \left[ \left( \mu_A - \frac{\sigma_A^2}{2} \right) T, \sigma_A \sqrt{T} \right]$$

and

$$\ln A_T \sim \phi \left[ \ln A_0 + \left( \mu_A - \frac{\sigma_A^2}{2} \right) T, \sigma_A \sqrt{T} \right]$$

or the asset value growth rate is normally distributed and, since the log of assets is normally distributed, assets follow a lognormal distribution. The methods used to simulate the asset diffusion model in Microsoft Excel are based primarily on the lectures of Dr. Dermot Hayes (2001) at Iowa State University. Recent work by Arnold and Henry (2003), discovered after the writing of this paper, discusses the technique in detail.

### **3.6.2 Correlated Returns**

To complete the simulation, it is also necessary to simulate the correlation between the value of the firm's assets and the level of the hedging index as well as an asset diffusion model for the index. The time zero realizations of index levels are found using the same diffusion model used to simulate the asset value diffusion. The random draws used to simulate the Weiner process for the index must be correlated to the random draw to simulate the Weiner process for the firm's asset values. The target correlation between the index and assets is calculated using historical values of asset values and index levels. In the simulation, the correlation between the random draw for the Weiner processes is adjusted until the correlation between the asset value and the index level is equal to the estimated historic correlation between the two variables.

### **3.7 Estimation of Asset Value and Asset Volatility**

In order to implement a hedging strategy under this framework, it is necessary to establish some characteristics of the assets of the firm. Specifically, we must establish historic values of assets from which we will estimate a correlation with the hedging instrument. The simplest method of assigning a value to the firm's assets would be to use an accounting (book value) definition of asset value. This would be some notion of net worth of the firm as represented on the firm's balance sheet. This, however, is an undesirable method to determine asset value as it is strictly an accounting definition which has the shortcoming of not representing the actual market value of the firm's assets nor is it a valid metric to use to determine the firm's ability to service its debt. To clarify this point, it is useful to use the definition of an asset as the discounted future cash flow associated with holding an instrument. It is not possible to determine this discounted value using accounting definitions.

Instead, we will use the market value and volatility of the firm's equity to determine the value of the firm's assets simultaneously with the volatility of the firm's assets.

Following a result developed by Jones, Mason, and Rosenfeld (1984) and further refined by Hull, Nelken, and White (2003), the following relationship between the market value of equity, volatility of equity, and the value and volatility of the firm's assets:

$$E\sigma_E = \frac{\partial E}{\partial A} A\sigma_A \quad (1)$$

where:

$E$  = time 0 value of the firm's equity,

$\sigma_E$  = the time 0 volatility of the firm's equity,

$A$  = the time 0 value of the firm's assets,

$\sigma_A$  = the time 0 volatility of the firm's assets.

$E$  is observed in equity market data and  $\sigma_E$  is estimated using implied volatility reported by Bloomberg.  $\partial E/\partial A$  can be represented as an option delta (Hull, Nelken, and White 2003).

To derive the specific option delta, first represent the value of the equity of the firm as a European option on the firm's value. As discussed previously, the firm can either be solvent or in default at time  $T$ . Again, assume that the firm issues an amount of debt with face value equal to  $FV$  at time 0. If, at time  $T$ ,  $A \geq FV$  then the debt holders are paid and amount  $FV$  and equity holders are left with  $A - FV$ . If, instead at time  $T$ ,  $A < FV$ , then the debt holders are paid an amount  $A$  and the equity holders are left with nothing. Therefore, the value of the firm's equity at time  $T$  is given by:

$$E_T = \max(A_T - FV, 0)$$

Thus, the value of the firm's equity is a call option on the value of the firm's assets with a strike price  $FV$ . To price the firm's equity at time 0, we simply apply the Black-Scholes-Merton option-pricing model to the call. Therefore, the value of the firm's equity at time 0 is given by:

$$E = A \times N(d_1) - FV e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(A/FV) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma_A \sqrt{T}}$$

$$d_2 = d_1 - \sigma_A \sqrt{T}$$

Notice that the delta of this call is given by:

$$\Delta_E = \frac{\partial E}{\partial A} = N(d_1)$$

We can now write equation (1) as:

$$E \sigma_E = N(d_1) A \sigma_A \quad (2)$$

Solving for  $A$ , this expression becomes:

$$A = \frac{E \sigma_E}{\sigma_A N(d_1)}$$

This expression is then substituted into the above equation to obtain:

$$E = \frac{E \sigma_E}{\sigma_A N(d_1)} [N(d_1) - L \times N(d_2)]$$

$$\Rightarrow 1 = \frac{\sigma_E}{\sigma_A N(d_1)} [N(d_1) - L \times N(d_2)]$$

where

$$d_1 = \frac{-\ln(L)}{\sigma_A \sqrt{T}} + .5 \sigma_A \sqrt{T}$$

$$d_2 = d_1 - \sigma_A \sqrt{T}$$



Thus, we can solve for  $\sigma_A$  using an iterative search technique on this equation. Notice that to carry out this calculation, it is necessary to use an estimate of leverage (L). Intel had virtually no debt at the time of the simulation, thus  $L = 0$ . We can then use equation (2) to solve for the value of the firm's assets at time 0.

Certainly, this method of asset valuation is not without potential peril. There are numerous recent examples in which the market as a whole has been duped by less than forthright accounting practices which will miss-price equity, and therefore, assets. The benefit of using this method, however, is imbedded in the premise that the market displays at least some degree of efficiency. While this method is obviously imperfect, we contend that it is at least as good as using pure accounting values. To create a time series of asset values, we used end of day equity prices from Yahoo and implied equity volatility data from Bloomberg for the two years ending November 20, 2003.

This method used to value the firm's assets at time 0 does not directly tell us anything about the future value of those assets. In order to carry out the simulation using the Merton bond-pricing model, we must assume that the value to the firm's assets follow a lognormal diffusion process. Under different bond pricing models, it may be possible to relax this assumption and use more sophisticated time series tools to model the value of the firm's assets over time.

### **3.8 The Hedge Ratio**

The hedging strategy that we will consider is a simple static full delta hedging strategy. The basic idea behind full delta hedging is to hold a position in an asset whose value moves in the opposite direction of the value of the variable that is being hedged. The quantity of the hedging instrument is chosen such that the magnitude of the movement in the

value of the hedged variable is the same as the magnitude of the movement in the value of the hedging instrument. In this example, we will simply purchase a put option on a stock index. To be an effective hedge, the option should pay off in the states of the world when the value of the firm's assets is low. Using an option on a stock index in this case is actually a cross hedge. A cross hedge is a hedge that uses an instrument whose value is not dependent on the same underlying variable as the variable that is being hedged. A hedge that would not be considered a cross hedge would be the purchase of a put option on the value of the firm's assets. Choosing the index option to use is simply a matter of finding the index most closely correlated with the variable that we wish to hedge. If there is only a weak correlation between the hedged variable and the index, the hedge will be ineffective as the values of the two assets move (almost) independently of one another. This is an example of basis risk. Not surprisingly, the NASDAQ 100 is the index that is most closely correlated of those tested to the asset value of Intel Corporation. The NASDAQ 100 tracks the 100 largest companies that trade on the NASDAQ exchange which is made primarily of technology companies. Since Intel is also a technology company, we expect movements in the asset value of Intel to closely track movements in the index value.

Here, we wish to hedge the value of the firm's assets, so we need to know how the value of  $A$  changes with the value of the index option,  $NDX$ . Then,

$$\Delta_h = \frac{dp_{NDX}}{dA} = \frac{dp_{NDX}}{dNDX} \frac{dNDX}{dA} = \Delta_{p_{NDX}} \frac{dNDX}{dA}$$

where  $\Delta_h$  is the number of "units" of the firm's assets (we use assets per share) held for one unit of the hedging instrument.  $p_{NDX}$  is the value of the index put used for hedging, and  $\Delta_{p_{NDX}}$  is the delta of the index option with respect to the value of the index.  $\Delta_{p_{NDX}}$  can easily

be found by taking the first derivative of the Black-Scholes put option formula with respect to  $NDX$  (Hull 2000):

$$p = S \exp(-rT)N(-d_2) - NDX \bullet N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{NDX}{S}\right) + (r + \sigma_{NDX}^2 / 2)T}{\sigma_{NDX} \sqrt{T}}$$

and

$$d_2 = d_1 - \sigma_{NDX} \sqrt{T}$$

and  $p$  is the time = 0 value of the put,  $S$  is the strike price of the put,  $N(\bullet)$  is the cumulative probability distribution function (cdf) of a standard normal distributed variable,  $NDX$  is the time = 0 value of the index,  $r$  is the risk free interest rate,  $T$  is the time to expiration, and  $\sigma_{NDX}$  is the volatility of the index value. Then, the derivative of the option value with respect to the value of the underlying index is given by:

$$\frac{dp_{NDX}}{dNDX} = \Delta_{p_{NDX}} = N(d_1) - 1$$

$NDX$  and  $T$  are easily observable.  $S$ , the strike price of the option is a choice variable for the firm. By choosing  $S$ , the firm can choose to hedge with an at-the-money, out-of-the money, or in-the-money put. We simulated each of the three cases. There are two options for calculating  $\sigma_{NDX}$  in the options pricing formula. First, we could use historical data to calculate an estimate of  $\sigma_{NDX}$ . This is not the preferred method of calculation, however, because  $\sigma_{NDX}$  is an instantaneous volatility. A historical estimation technique does not tell us what the volatility is at the current time. To remedy this, we instead use the implied volatility of the option as an estimate of  $\sigma_{NDX}$  as reported by Bloomberg. The estimate of  $\Delta_{p_{NDX}}$  is -0.83.

Unfortunately, we are forced to rely on historic data to calculate  $dNDX/dA$ . This ratio can be determined by running the following OLS regression:

$$NDX_t = \alpha + \beta A_t + \varepsilon_t$$

We then use  $\hat{\beta} = \sigma_{A,NDX} / \sigma_A^2$  as an estimate of  $dNDX/dA$ . In the simulation, the values of NDX are daily values of the NASDAQ 100 index for the two years ending November 20, 2003. Daily asset values are calculating using Hull's method as described earlier. For the base case, our estimated beta from the regression is 31.60. Also from this regression, we take the square root of the  $R^2$  of the regression to obtain an estimate of correlation coefficient between the value of the assets and the index. The estimated correlation was 0.9572. The hedging delta is then equal to  $(-0.813)(31.6) = -25.70$ . This delta is then negated and inverted (0.0389) to find the number of put options per share of stock that must be purchased to place the hedge.

## 4. SIMULATION ESTIMATES

### 4.1 The Base Case Simulation

The first simulation (the base case simulation), that we ran use the estimates of the parameters from the firm's market data using an at-the-market put option as the hedging instrument (*Table A*). As expected, the average cost of debt (bond yield) under the hedge is considerably less than the average cost of debt for the unhedged case. Of course, this result is driven by the decrease in variance of time zero asset value. As expected, the time zero mean value of assets in the hedged and unhedged cases are almost the same (unhedged 65.77 and hedged 65.91). Furthermore, there is a significant decrease in the standard deviation of the time zero asset value (unhedged 21.38 and hedged 15.71). A histogram of the realizations of asset values for the unhedged case is represented in *Figure B*. The hedged asset value distribution is shown in *Figure C*, and both distributions are shown together in outline in *Figure D* for comparison. Just as suggested by the descriptive statistics of the distributions, there is much less dispersion in the hedged case.

Because the yield function is convex in asset value, it is necessarily true that the average cost of debt will be lower when the standard deviation of asset value decreases. For the base case simulation, the average discounted bond yield decreased from 1.878% for the unhedged case to 0.8745% for the hedged case. The spread between the unhedged yield and the hedged yield is 1.003%. The ratio of the unhedged yield to the hedged yield is 2.15. By failing to hedge, the firm more than doubles its financing cost. An added benefit of hedging is that the variance of the discounted bond yield will be lower than in the unhedged case, making it easier to forecast future yields. For the hedged case, the standard deviation of the yield is 0.00793488. For the unhedged case, the standard deviation of the yield is 0.02611464. The ratio of the unhedged standard deviation to the hedged standard deviation

is 3.29. A histogram the unhedged yield realizations is shown in *Figure E*. The hedged distribution is shown in *Figure F*, and both distributions are shown together in *Figures G and H*.

The histograms give some insight into the benefit of hedging. As can be seen in *Figure G*, hedging decreases the frequency of extremely low bond yields. This occurs because of the outlay of money required to place the hedge. In the cases where the realization of asset value is high, the level of the index will be high as well. Therefore, the put used to hedge will not pay off and the asset value will be decreased by the time -1 cost of the hedge. On the other hand, *Figure H* also shows the source of the benefit of the hedge. *Figure H* is the right tail of the two distributions. The hedged distribution truncates before the unhedged distribution. It is clear that, for this particular hedging strategy, the decrease in the average cost of debt is a result of the diminished probability of extremely poor bond yield outcomes

Another metric included in each table reporting the simulation results is the per share increase in cash flow (per share cost reduction) given the hedge. The per share cost reduction is calculated by multiplying the spread between the hedged and unhedged bond yields by the amount of borrowing. This cost reduction metric is identical to the representation of cost in the cash flow model and is equal to the following:

$$e\left(\frac{P^f}{P_u^b} - 1\right) - e\left(\frac{P^f}{P_h^b} - 1\right)$$

where  $P_u^b$  is the price of the bond in the unhedged case and  $P_h^b$  is the price of the bond in the hedged case. In the base case, this per share cost reduction is approximately \$0.33, which is

a substantial increase in cash flow, particularly if that increase continues for every period into the future.

It is worth noting at this point that an extension of the model may lead to even greater decreases in the cost of debt. If we were to either extend the hedge to time one or place a new hedge at time zero that pays off at time one, we would decrease the expected volatility of the firm's assets between time zero and time one. This, in turn, would decrease the value of the put option portion of the bond valuation formula, thus increasing the price of the bond and decreasing the bond yield. There may be a credibility problem with this addition to the hedging strategy as the firm has no incentive to hedge once the bonds are sold. The firm, however, does not have any particular incentive not to hedge as the hedge, on average, is costless.

#### **4.2 Strike Price of the Hedge**

We ran another set of simulations in which we varied the strike price of the put option used to hedge. *Table B* shows the outcomes of these simulations. The first row represents the base case simulation using an at-the-money put option. The next two rows show the results for in-the-money puts. For example, the second line shows the outcome for the put with a strike price that is 125% of the time zero value of the NASDAQ 100 index. The last line of the table shows the outcome for a put strike price that is only 90% of the time zero value of the index.

As can be seen in *Table B*, there is little difference between the average bond yield generated with the at-the-money put and varying levels of in-the-money puts. This is to be expected as all of those options have almost the same net payout (net of the cost of the option) for any downward movements in the index. The at-the-money put will not payoff in

a few situations when the in-the-money puts do payoff, so we expect the yield for the at-the-money case to be slightly greater than the in-the-money cases. The out-of-the-money put is less effective in decreasing the average bond yield because the option does not pay off for smaller downward movements in the index value. In other words, there is a minimum decrease in value in the index that must occur before the hedge become effective.

### 4.3 Debt Level

The next group of simulations that we ran is for different amounts of borrowing. As can be seen in *Table C*, the benefit of hedging increases as the amount of debt increases over the range studied. Here, the debt ratio is measured as the ratio of debt to the total value (debt and equity) of the firm. As suggested earlier, the hedge is of little benefit for very low levels of debt. In fact, at the 5% debt level, the spread between the unhedged yield and the hedged yield is less than  $1\% \times 10^{-6}$  and the total discounted cost reduction is only about  $\$1 \times 10^{-8}$  per share. The positive relationship between level of debt and the yield spread suggests that, for higher levels of debt, the yield function is more convex in asset value. We will not examine this relationship analytically, but, as can be seen in *Figure I*, the yield curve for the 75% debt level appears to be more convex than the yield curve for the 50% debt level for most of the range over which we have observations.

### 4.4 Risk Free Rate

Finally, we ran simulations with a number of different risk free interest rates. As can be seen in *Table D*, it is difficult to specify the effect of higher risk free interest rates on the yield spread, but there is a positive relationship between the bond yield and the risk free rate. Intuitively, there are a number of things that occur with the increase in the risk free rate (Hull



2000), but for our simulation, the important affect is on the payoff of the hedging option.

The expected growth rate of the index value increases. It then follows that the index option does not pay off quite as often for a higher risk free rate when an at-the-money option is used to hedge, even for negative changes in asset value in some realizations. Obviously, this effect is not very great.

## 5. CONCLUSION

We have shown that hedging is warranted in many cases based only on the savings in the cost of debt. The magnitude of the bond yield decrease when the hedge is undertaken depends on a number of factors such as the strike price of the option used and the level of debt issued by the firm. As hypothesized, however, we found certain conditions under which the benefit to hedging is minimal such as when the amount borrowed is very small. Therefore, firms that have little chance of requiring external finance, do not have a motivation to hedge.

While hedging to reduce the cost of debt is promising, there is another benefit of hedging that we were not able to explore in the simulation, but which add value to the firm. As suggest by Froot, Scharfstein, and Stein, the firm is able to increase investment spending by hedging cash flow. Cash flow hedging is, in fact, what we have done in our simulation. We need only to be able to estimate cash flow and a production function in conjunction with the estimation that we have already done here.

While our research did not uncover any evidence that this type of hedging is prevalent in practice, it is difficult to know exactly what types of hedges firms use because of the lack of reporting of financial derivatives transactions. The size of the derivatives market is growing, but that does not tell us exactly how those derivatives are being used. Studies by Allayannis and Mozumdar (2000) and by Adam (2002) offer support to the use of hedging to reduce dependence on expensive external financing. What remains to be explored is the extent to which hedging firms seek to reduce the unit cost of debt. As we increase our understanding of hedging and firms are able to implement transparent and effective hedging strategies, this type of cash flow hedging should become more widespread.

### Appendix A: The FSS model

The expected net present value of the level of investment of the firm,  $I$ , is given by the following:

$$f(I) - I$$

where  $f(I)$  is positively sloped and concave in  $I$ . The discount rate is set at zero for simplicity.

Investment expenditure is funded first by internally generated funds,  $w$ , and then by borrowed funds,  $e$ . Thus,

$$I = w + e$$

Since the discount rate is zero, creditors only require a repayment of  $e$  at time 1.

Furthermore, there is a deadweight cost associated with external finance given by  $C(e)$  which is convex in  $e$ . Therefore, profit is given by the following:

$$P(w) = f(I) - I - C(e)$$

The first order condition of profit maximization is:

$$f_I - 1 = C_e$$

The first order condition implies that there is underinvestment when external financing is costly. The authors then show that the second derivative of profit with respect to liquid assets is given by the following:

$$P_{ww} = f_{II} \frac{dI}{dw}$$

Thus, hedging is justified (profit is concave in cash) if the production function is concave in the level of investment and the effect of additional cash on the optimal level of investment is positive.

## Appendix B: Derivation of the Distribution of Asset Values

In this appendix, we will derive the properties of the distribution of asset value at time zero. The derivation follows closely from Hull (2000). First, we assume that the change in the value of assets follows Geometric Brownian Motion:

$$dA = \mu_A A dt + \sigma_A A dz$$

where  $dz$  is a Wiener process. By Ito's Lemma (see Hull (2000) for a complete proof), we have the following:

$$dG = \left( \frac{\partial G}{\partial A} \mu_A A + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial A^2} \sigma_A^2 A^2 \right) dt + \frac{\partial G}{\partial A} \sigma_A A dz$$

Then, let  $G = \ln A$ . We can then solve for the parameters of the above equation as follows:

$$\begin{aligned} \frac{\partial G}{\partial A} &= \frac{1}{A} \\ \frac{\partial^2 G}{\partial A^2} &= -\frac{1}{A^2} \\ \frac{\partial G}{\partial t} &= 0 \end{aligned}$$

It then follows that

$$dG = \ln A = \left( \mu_A - \frac{\sigma_A^2}{2} \right) dt + \sigma_A dz .$$

## Appendix C: Tables and Figures

Table A: Base case parameter values

Parameter	Estimate
$\rho_{A,NDX}$	0.957181
$r_f$	0.0122
$\sigma_A$	0.352157
$\sigma_{NDX}$	0.2894
NDX Option Price	11.13
$NDX_{-1}$	1363.49
$A_{-1}$	32.89
$\frac{\partial NDX}{\partial A}$	31.5973

Table B: Simulation results for varying hedge instrument strike prices.

In/Out of the Money	Strike	Hedged Yield	Unhedged Yield	Spread	Per Share Cost Reduction
At(Base)	1363.49	0.00874488	0.01877637	0.01003149	0.32995084
25% in	1704.3625	0.00808323	0.01877637	0.01069314	0.35171356
50% in	2045.235	0.00845335	0.01877637	0.01032302	0.3395398
10% out	1227.141	0.01314693	0.01877637	0.00562944	0.18516086

Table C: Simulation results for varying levels of debt.

Debt Ratio	Hedged Yield	Unhedged Yield	Spread	Per Share Cost Reduction
5%	2.3154E-12	6.7375E-08	6.7373E-08	1.1663E-07
25%	8.9519E-05	0.00110641	0.00101689	0.01114904
50%	0.00874488	0.01877637	0.01003149	0.32995084
75%	0.07895115	0.10064967	0.02169852	2.14108927

Table D: Simulation results for varying risk free rates.

Risk Free Rate	Hedged Yield	Unhedged Yield	Spread	Per Share Cost Reduction
2.5%	0.00875354	0.01881202	0.01005848	0.33083857
5%	0.0089589	0.01895402	0.00999512	0.32875469
10%	0.01092497	0.01954608	0.00862111	0.28356138



## Corporate Bond Payoff Diagram

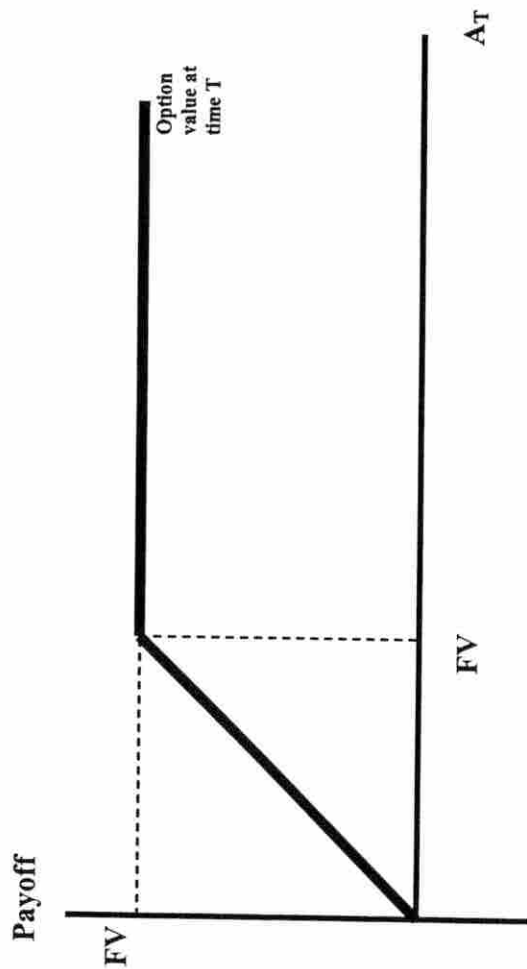


Figure A: The payoff diagram for a zero-coupon bond at maturity. For asset values less than the face value of the bond, the payoff is equal to the value of the assets. For asset values greater than the face value of the bond, the payoff is equal to the face value.

# Unhedged Asset Value Distribution

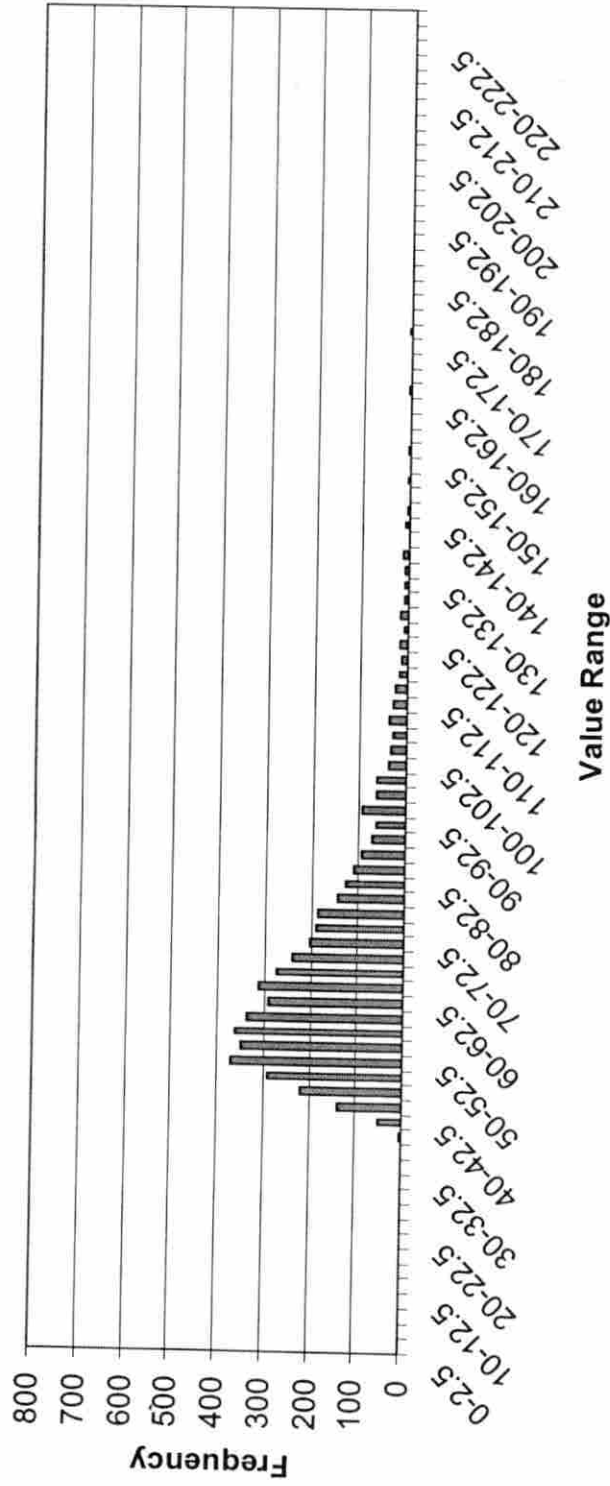


Figure B: The distribution of the time zero asset value when the firm does not hedge.

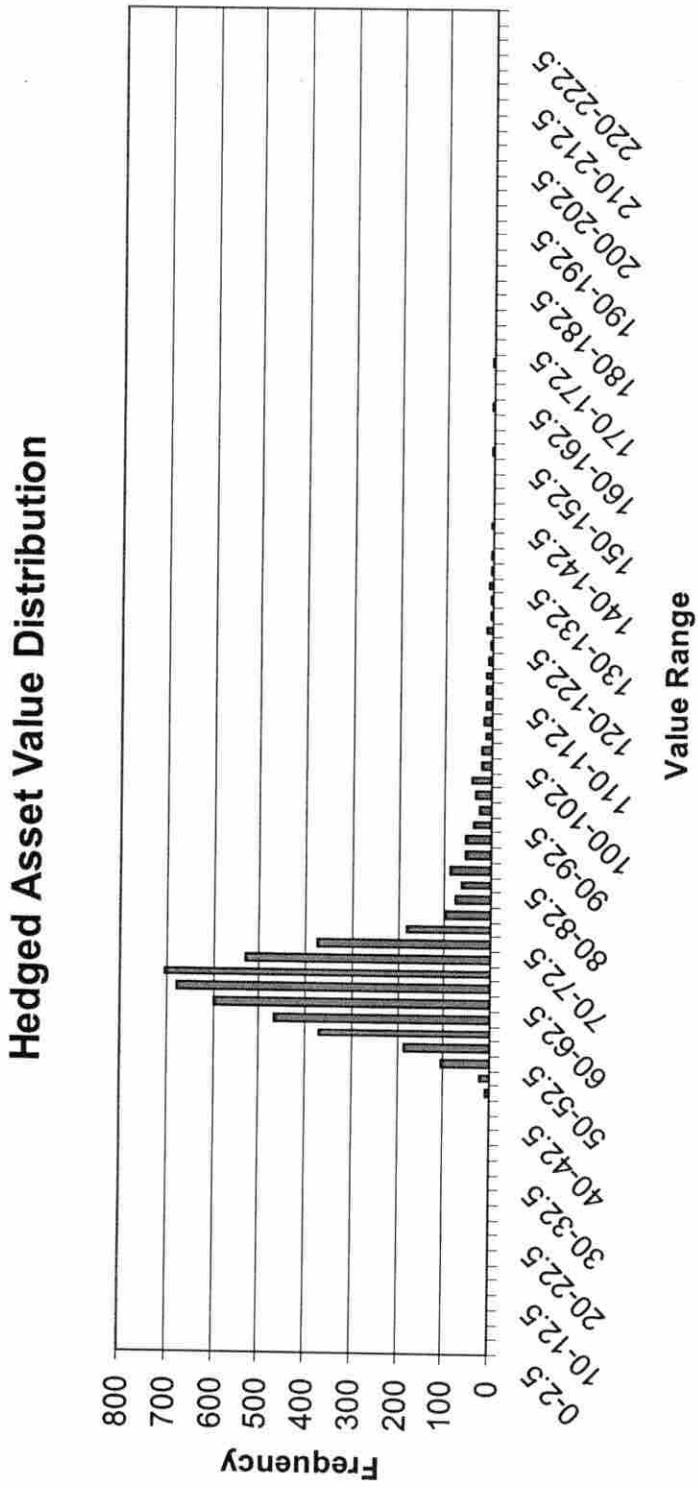


Figure C: The distribution of the time zero asset value when the firm does hedge.

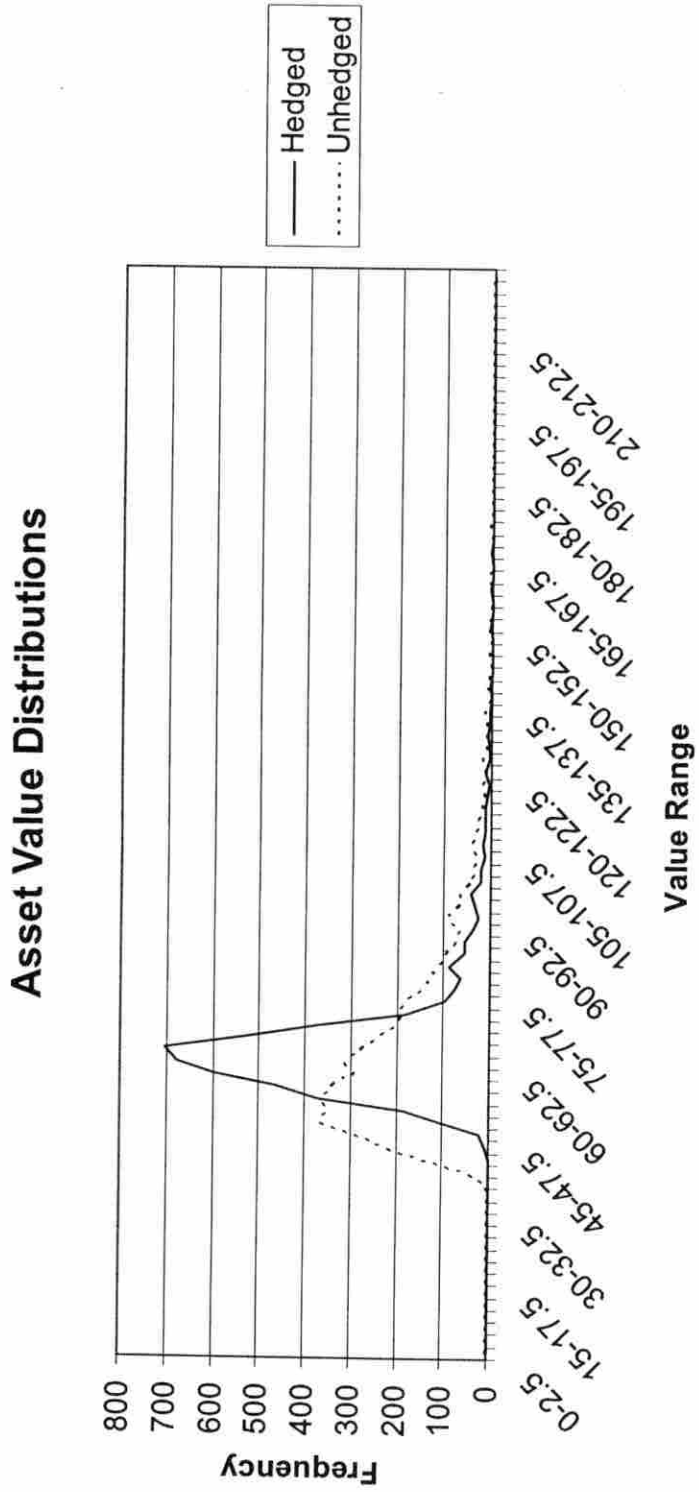


Figure D: The distributions of the time zero asset value when the firm does hedge and does not hedge.

# Unhedged Yield Distribution

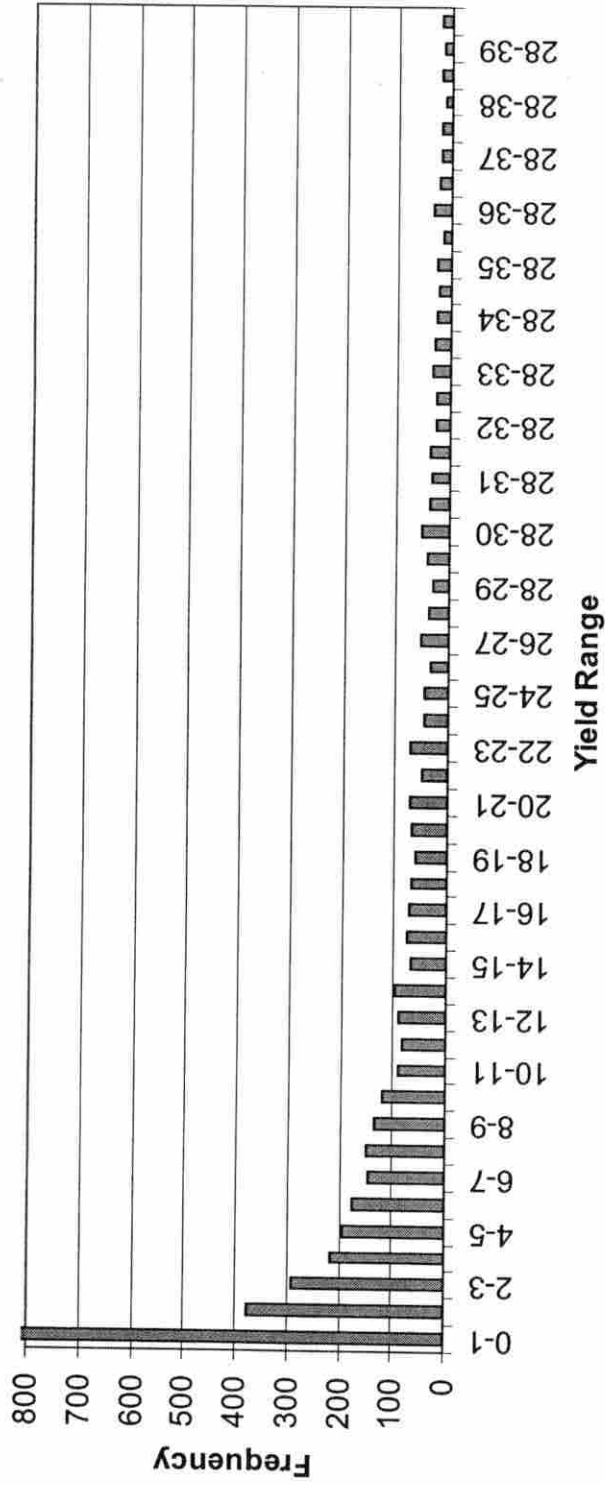


Figure E: The distribution of the time zero bond yield when the firm does not hedge. Yields are represented as percentages.

## Hedged Yield Distribution

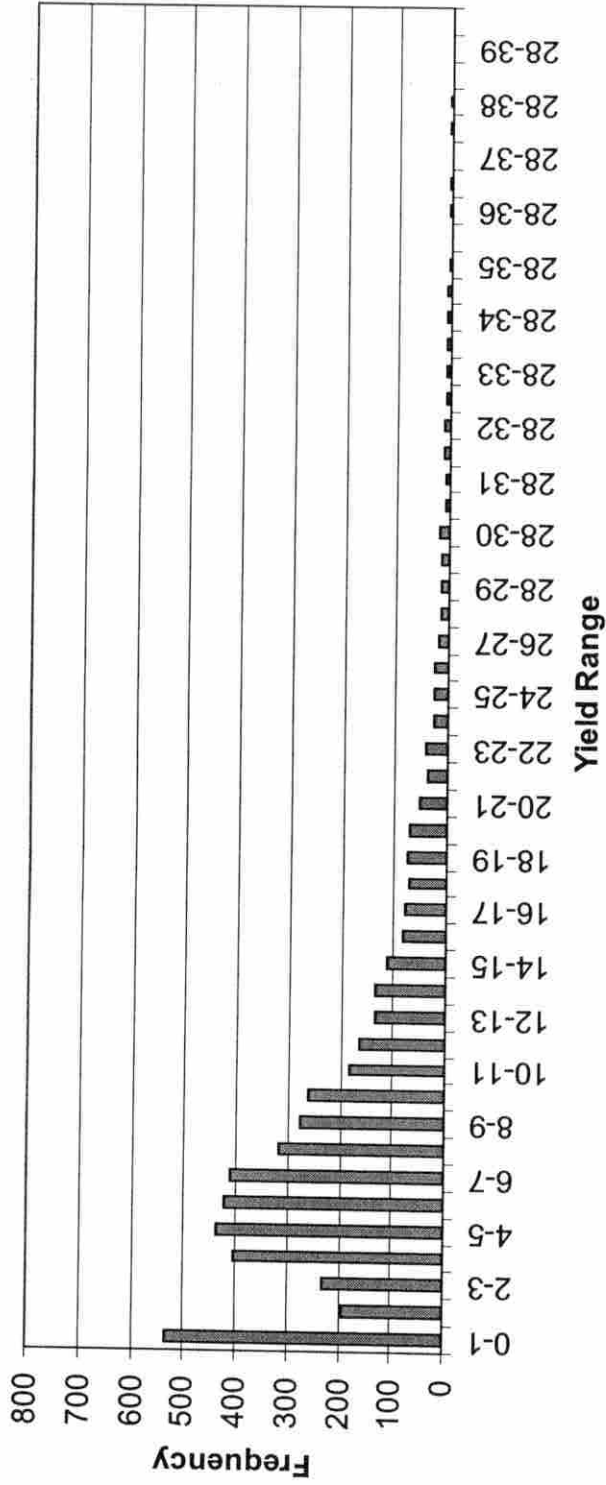


Figure F: The distribution of the time zero bond yield when the firm does hedge. Yields are represented as percentages.

## Yield Distributions

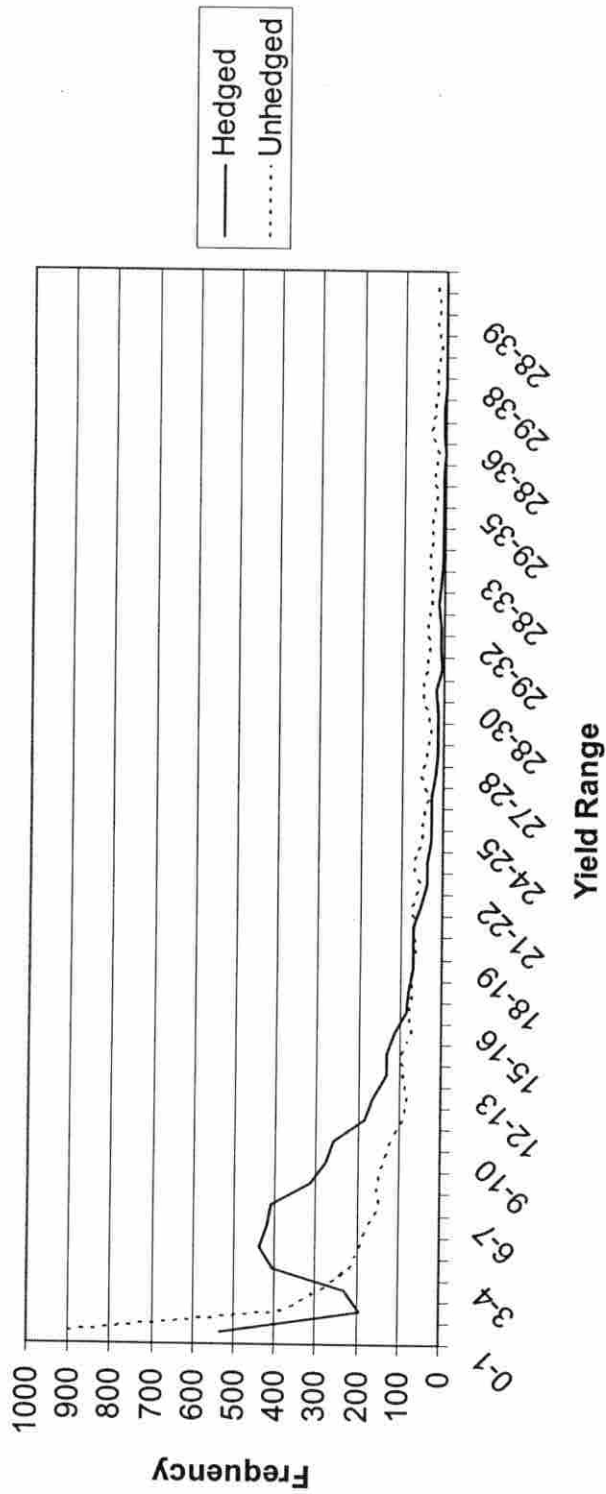


Figure G: The distributions of the time zero bond yield when the firm does hedge and when it does not hedge. Yields are represented as percentages.

# Yield Distributions

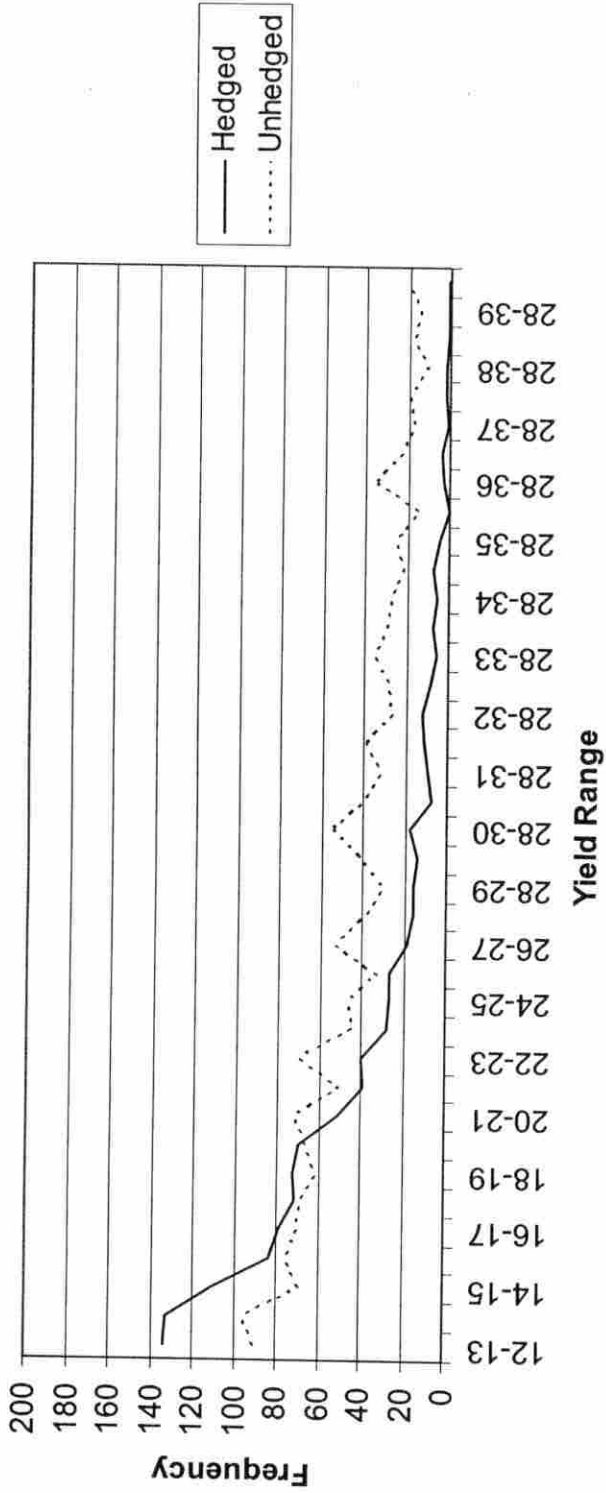


Figure H: The right-hand tail of the distributions of the time zero bond yield when the firm does hedge and when it does not hedge. Yields are represented as percentages.



## Yield Curves

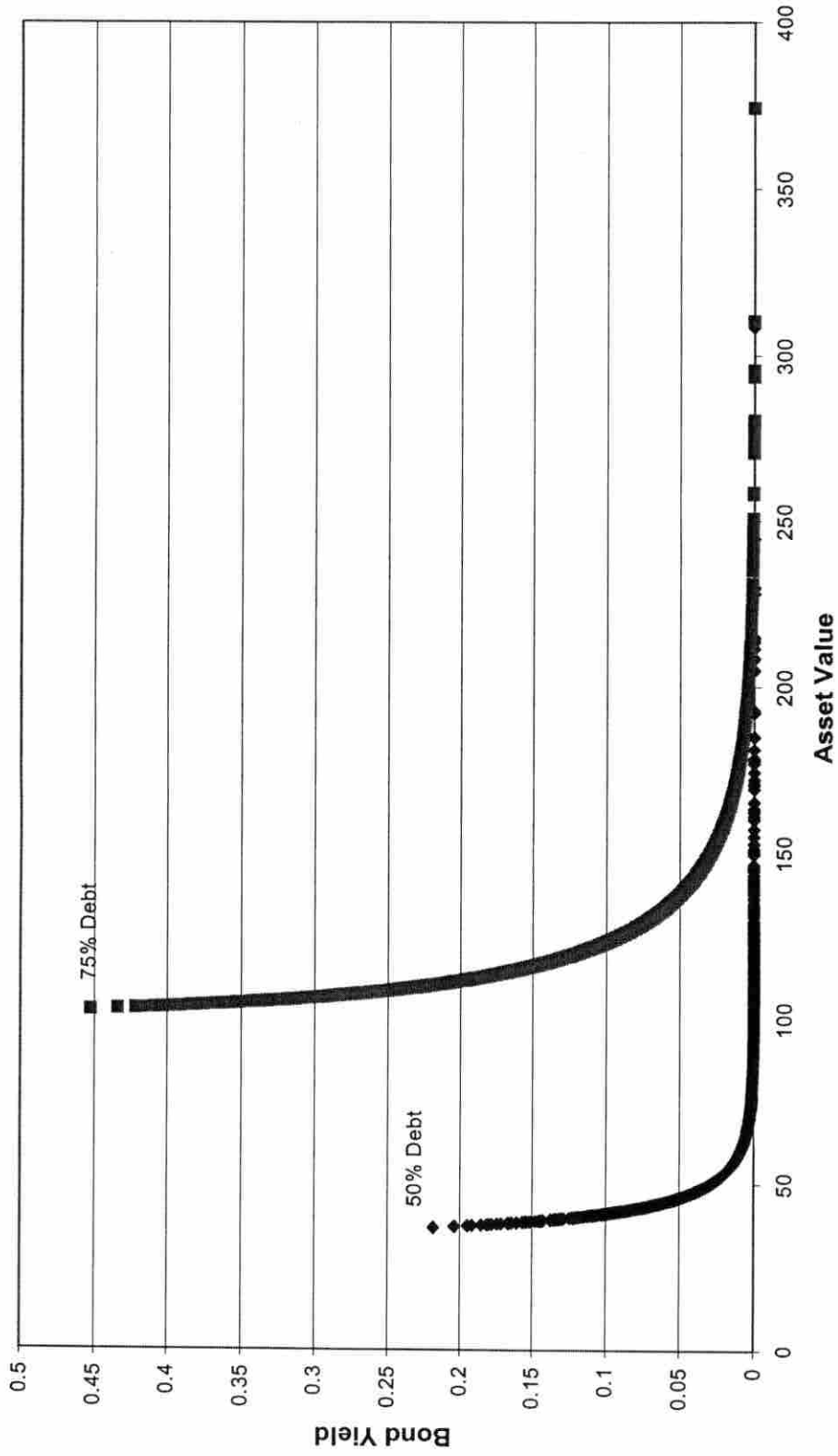


Figure 1: The yield curves (as a function of asset value) for a firm with a 50% debt to value ratio and a 75% debt to value ratio.

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